Case III $\Rightarrow$ $D = 0$ i.e one root of λ - cubic is zero and $ul_3 + vm_3 + wn_3 = 0$ where l_3, m_3, n_3 are actual direction cosines of axis corresponding to $\lambda = \lambda_3 = 0$, then $f(x, y, z) = 0$ reduces to either of the following three forms :

$Ax^2 + By^2 + C = 0$ (Elliptic cylinder)

$Ax^2 - By^2 + C = 0$ (Hyperbolic cylinder)

$Ax^2 - By^2 = 0$

Ex - Show that the surface represented by

$$x^2 + 6y^2 - z^2 - yz + 5xy + 2x + 5y = 0$$

represents hyperbolic cylinder.

→ The given equation -

$$x^2 + 6y^2 - z^2 - yz + 5xy + 2x + 5y = 0$$

Comparing it with.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

$a = 1$, $f = \frac{-1}{2}$, $u = 1$, $d = 0$

$b = 6$, $g = 0$, $v = \frac{5}{2}$

$c = -1$, $h = \frac{5}{2}$, $w = 0$

$A = bc - f^2$

$= (6)(-1) - \left(\frac{-1}{2}\right)^2$

$= -6 - \frac{1}{4}$

$= \frac{-24-1}{4}$

$= -\frac{25}{4}$

$B = ca - g^2$

$= (-1)(1) - 0$

$= -1$

$D = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (1)(6)(-1) + 2\left(\frac{-1}{2}\right)(0)\left(\frac{5}{2}\right) - (1)\left(\frac{-1}{2}\right)^2 - (6)(0) - (-1)\left(\frac{5}{2}\right)^2$

$= -6 - \frac{1}{4} + \frac{25}{4}$

$= \frac{-24-1}{4} + \frac{25}{4}$

$= -\frac{25}{4} + \frac{25}{4}$

$= 0$

The discriminating cubic is

$$\lambda^3 - (a+b+c)\lambda^2 + (A+B+C)\lambda - D = 0$$

$$\lambda^3 - (1+6-1)\lambda^2 + \left(\frac{-25}{4} - 1 - \frac{1}{4}\right)\lambda - 0 = 0$$

$$\lambda^3 - 6\lambda^2 + \left(\frac{-25}{4} - \frac{4-1}{4}\right)\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + \left(\frac{-25}{4} - \frac{5}{4}\right)\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + \left(-\frac{30}{4}\right)\lambda = 0$$

$$\lambda^3 - 6\lambda^2 - \frac{15}{2}\lambda = 0$$

$$\lambda\left(\lambda^2 - 6\lambda - \frac{15}{2}\right) = 0$$

$$\lambda(2\lambda^2 - 12\lambda - 15) = 0$$

$$\lambda = 0, \quad \frac{12 \pm \sqrt{144 - 4(2)(-15)}}{4}$$

$$\lambda = 0, \quad \frac{12 \pm \sqrt{144 + 120}}{4}$$

$\lambda = 0, \dfrac{12 \pm \sqrt{264}}{4}$

$\lambda = 0, \dfrac{12 \pm 2\sqrt{66}}{4}$

$\lambda = 0, \dfrac{12}{4} \pm \dfrac{2\sqrt{66}}{4}$

$\lambda = 0, 3 \pm \dfrac{\sqrt{66}}{2}$

$\lambda = 0, 3 \pm \dfrac{\sqrt{2 \times 33}}{2}$

$\lambda = 0, 3 \pm \dfrac{\cancel{\sqrt{2}} \times \sqrt{33}}{\cancel{\sqrt{2}} \times \sqrt{2}}$

$\lambda = 0, 3 \pm \dfrac{\sqrt{33}}{\sqrt{2}}$

$\lambda = 0, 3 \pm \sqrt{\dfrac{33}{2}}$

The direction cosines of the principal axis corresponding to $\lambda = 0$ are given by.

$\Rightarrow al + hm + gn = 0$

$l + \frac{5}{2}m + 0 = 0$

$2l + 5m = 0$

$\Rightarrow hl + bm + fn = 0$

$\frac{5}{2}l + 6m - \frac{1}{2}n = 0$

$5l + 12m - n = 0$

$\Rightarrow gl + fm + cn = 0$

$0 + \left(\frac{-1}{2}\right)m + (-1)n = 0$

$-\frac{1}{2}m - n = 0$

$-m - 2n = 0$

Solving first two equations, we get

$$\frac{l}{-5-0} = \frac{m}{0+2} = \frac{n}{24-25}$$

$$\frac{l}{-5} = \frac{m}{2} = \frac{n}{-1}$$

$$\frac{l}{5} = \frac{m}{-2} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{25+4+1}}$$

$$= \frac{1}{\sqrt{30}}$$

$l = \frac{5}{\sqrt{30}}$, $m = \frac{-2}{\sqrt{30}}$, $n = \frac{1}{\sqrt{30}}$

Now, $ul + vm + wn$

$= (1)\left(\frac{5}{\sqrt{30}}\right) + \left(\frac{5}{2}\right)\left(\frac{-2}{\sqrt{30}}\right) + (0)\left(\frac{1}{\sqrt{30}}\right)$

$= \frac{5}{\sqrt{30}} - \frac{5}{\sqrt{30}} = 0$

In this case, there is line of centres given by

$\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial z} = 0$

$2x + 5y + 2 = 0$

$5x + 12y - z + 5 = 0$

$y + 2z = 0$

Second and third equations give first equation

we solve first and third equations

Putting $z = 0$ in these two eqns

$2x + 5y + 2 = 0$

$y = 0$

$2x + 2 = 0$

$$2x = -2$$

$$x = -1$$

$\therefore$ any pt. on line of centres is $(-1, 0, 0)$

Also $d' = (1)(-1) + \left(\frac{5}{2}\right)(0) + (0)(0)$

$$= -1$$

Transfering the origin to $(-1, 0, 0)$ and turning the axes, the reduced eqn is

$$\lambda_1 x^2 + \lambda_2 y^2 + d' = 0$$

$$\left(3 + \sqrt{\frac{33}{2}}\right) x^2 + \left(3 - \sqrt{\frac{33}{2}}\right) y^2 - 1 = 0$$

which is hyperbolic cylinder

Axis passes through $(-1, 0, 0)$ and has direction ratios $\langle 5, -2, 1 \rangle$

its equation is $\frac{x+1}{5} = \frac{y-0}{-2} = \frac{z-0}{1}$